THE

MATHEMATICAL

GAZETTE

EDITED BY
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WITH THE CO-OPERATION OF
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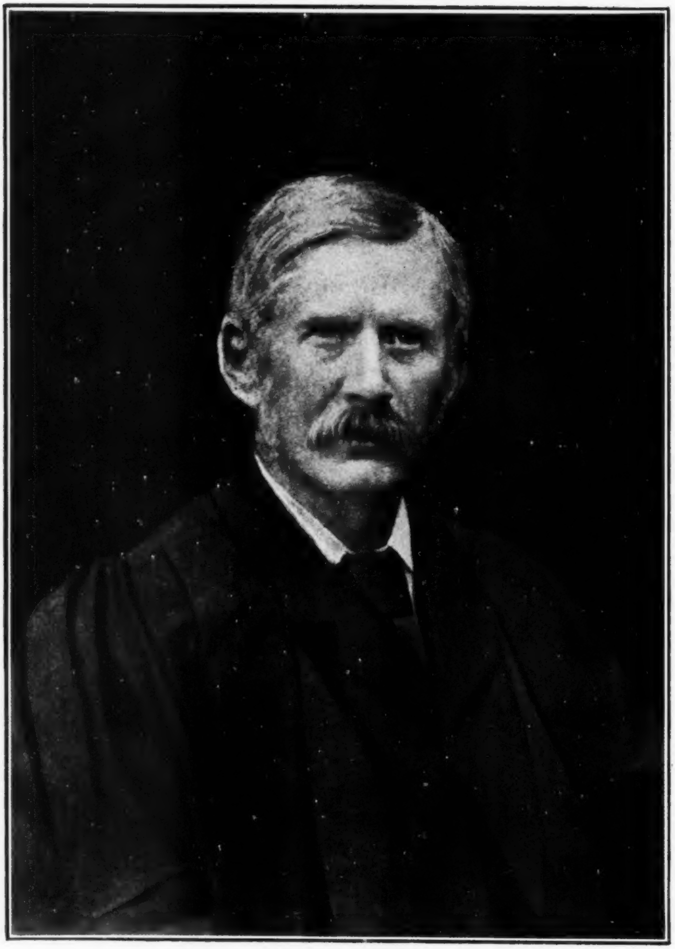

RAWDON LEVETT.

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A GREAT SCHOOLMASTER.

RAWDON LEVETT was, as was well said in the *Times*, a schoolmaster of genius; he was among the greatest teachers of his time. He died on February 1st at his home in Colwyn Bay at the age of seventy-nine.

After taking his degree as Eleventh Wrangler in 1865, he went for a short time to Rossall, but soon left to become mathematical master at King Edward's School, Birmingham, and here he did his life's work. He became Second Master, and refused for health reasons the Head Mastership which was pressed upon him. He retired from his work about twenty years ago. With him passed away the last of a great triumvirate: Vardy, Hunter Smith, Levett, masters of outstanding ability who not only made their mark upon the great school—where Lightfoot, Westcott and Benson had been taught—but added richly to the noble inheritance of honour which they had received.

Levett's name was very widely known to the last generation through the graceful words with which John Henry Shorthouse dedicated *John Inglesant* to him:

MY DEAR LEVETT.
I dedicate the volume to you that I may have an opportunity of calling myself your friend. J. HENRY SHORTHOUSE.
1880.

John Inglesant has now become a classic, and as such is read in many an Upper Form, but Mr. Levett's work is the more abiding, and is

As the sweet presence of a good diffused,
And in diffusion ever more intense.

It is difficult to measure the ever widening influence of a great schoolmaster; testimony to it is one of the profoundest and most far-reaching influences in our national life: and how inspiring was Levett's influence is borne from the fact that Shorthouse singled him out from among his many friends to dedicate his first and greatest work to him.

And what was it that made him so great? Just a gift, the wonderful gift of personality—the only educating factor in any schoolmaster. Organisation may be good, material aids may be good, but the only thing which makes a

lasting impression and which lifts the teaching in any class-room into a plane above the average is the personality of the teacher. The teacher is the school for the highest purposes of education. It is the memory of him, not of what he said or did, but of what he was that is the source of life and inspiration in his pupils.

When Levett went to King Edward's School, the mathematics were at a low ebb. There were some 400 boys, and he was the only mathematician on the staff; other men taught mathematics, but of them the first part of the famous saying was true, "I know nothing of mathematics; I never even taught it." Levett so arranged that all work other than arithmetic was begun by himself: he initiated boys into algebra and geometry: from the very first he laid the utmost stress upon the doing of geometrical riders,—with all his powers he encouraged boys to make that initial effort, after which all other efforts are comparatively easy, to do their first rider, and how delightful was his pleasure in their success. He had, however, to create an atmosphere in which the growth of mathematics could be fostered. It was a severely classical school into which he had come, proud of its tradition for classical scholarship, its records would bear comparison with those of any school—five of its old pupils had not very long before been Senior Classics in six consecutive years. There were only some sixty boys learning any mathematics other than arithmetic, and only one now and then doing anything more than the merest elements of algebra and geometry. When he left a large part of the school was doing mathematics of an advanced type, year after year his old pupils had been gaining Trinity and other scholarships, the last two Senior Wranglers were King Edward's School boys. But he would have been the very last to set himself to improve his side of the school work at the expense of any other; he believed in mathematics as an instrument of education destined to uplift the intellectual standard of every side of school work. He scorned the examination test, and poured out upon it some of the expressions of his greatest contempt—but of that a little later—though he was not at all averse at first from making use of the stimulus of an examination to create an interest in his subject.

In his early days he persuaded the then Head Master to institute an arithmetic examination for the whole school. He set two graduated papers for the 400 boys. About a week after the examination the result in order of merit was read out to the whole school assembled in the Great School Room. This stimulated interest and keenness, and gradually fostered that growth which he set himself to bring about. Later on, when he had a mathematical staff and a mathematical organisation, this examination was dropped; but it will be interesting to mathematical readers to know that from the first his papers in arithmetic were divided into two distinct parts, the first part was the ordinary type of paper, the second the longer and more important part upon which he laid great stress, consisted entirely of questions on fundamental principles; reasons, explanations were asked for, there were no sums to be worked out. Arithmetic was regarded as a science depending upon law. I have never come across other papers quite like these.

As time went on his influence did its work—he won over the Governors to his side, he obtained a mathematical staff, and the school was fully organised for mathematical teaching; and thus far he knew at last that some of his work was done. One of the things he asked me a short time before his death was about the organisation at the old school; there was an anxious look in his piercing eyes as he turned to ask the question, and a wonderful relief as I assured him that nothing was altered, the work which he had begun was still being carried on with the same ideals as he had left it.

And what made the effect of his teaching so deep and lasting was the impression he left upon his pupils that they were his fellow-workers and must help him: that it was "not what a man does which exalts him, but what he

aspires to do." Endeavour to him was everything, success was naught—though success came. He encouraged every effort with a joy which was contagious; and even though he might have to give the same explanation over and over again, he would say, "You are quite right, do not leave this difficulty till you understand." I have already said with what scorn he regarded work merely for an examination—if a thing was worth doing it was worth doing for its own sake, it is work for work's sake that is needed; he used to refer contemptuously to that type of work "which collects knowledge to pour it down the sink of an examination"; he would have nothing to do with such special classes as the Army class, etc.—organisations which were to him the absolute negation of all true educational ideals. On one occasion he asked a boy at the beginning of the September term if he would care to revise for his Trinity Scholarship Examination in the following December; the boy's ready answer, "No, I will take it in my stride," delighted him. It was Levett's ideal. It illustrates the confidence which his teaching had inspired.

When he had been a short time in Birmingham, the Head Master told him that he wished the school to be examined by examiners from Oxford and Cambridge, and asked him to select some Forms to send in. Levett said he proposed to send in the first, the middle and the last boy from every Form; "this would give a correct impression of the work which was being done"; the Head Master replied that was not at all what he wished; and Levett definitely refused to have anything to do with the selection of boys, and in the contest between the two the Head Master did not win.

This is not the place to do more than mention that everyone of the varied school activities was deeply indebted to him: he started the School Club which controls the games, the sports, the debating, the music, the literary and other societies, and was its president for many years. His was the master spirit which led to the purchase of the playing fields. He contributed liberally to every effort which would broaden outlook and deepen the sense of *esprit de corps*. Eventually the fortune which he left is to go to help the scholarship and games of the school which he loved. I need not refer to the great work he did in connection with the Old Boys' Club; its marked success is largely due to him, to his sympathy, encouragement and guidance. It is destined to play an important part in the wider life of the city. With all this he had time for outside interests, and not only in Birmingham; he was one of the founders of the Association for the Improvement of Geometrical Teaching—now the Mathematical Association—he was its secretary for many years, the difficult years of its early growth—later on he became a Vice-President.

But to return to King Edward's School. His influence over the whole school was profound, and to those who knew him well he has been the spring of life. He wanted his boys to be true, to live true lives; he tried to make them feel the dignity of work and the responsibility of power. He saw in a great school placed in the centre of a great city a unique opportunity, not merely for the imparting of information, which according to Bishop Butler is the least part of education, but for the uplifting of the whole of the corporate life; he saw that from it could come the realisation that true education is a spiritual activity, without which rich men are really poor, and that this activity must find its expression in the translation of energy from one noble purpose to another, and that in faithful labour, done to the best of one's powers, lies the chief joy and the chief power in life.

I cannot conclude this brief record without quoting from the testimonials which his colleagues and old pupils gave him when he retired.

The former, in a farewell letter accompanying a beautiful gift, write:

"For your successful efforts in the promotion of scholarship, your beneficial influence on the growth of character, your judicious encouragement of healthful pastimes, your pupils of all ages and conditions have ever eagerly accorded you the tribute of their affectionate regard.

"We, for our part, are glad to acknowledge our especial obligation to you as an example of qualities, which, essential as they are to the genius of the teacher, do not lie within the compass of every schoolmaster. Devotion to duty, unselfishness of purpose, a frank and kindly humour, an unfailing sense of justice—these qualities harmonising with a generous and cultured humanity, have, throughout your career, dignified and sweetened your association with your fellow-workers, and have, moreover, unobtrusively afforded a salutary stimulus to their zeal in the cause of education."

The old boys collected a large sum of money for a gift for him, but at his wish it was spent on the School. I will quote from the letter which was signed by many hundreds:

"We cannot easily express our sense of all that we owe to you, and of the loss which the School sustains by your withdrawal from its daily life.

"Those of us who were your pupils knew you as an ideal teacher, patient with all our weaknesses, and stimulating all our powers, opening to us constantly new regions of interest and setting before us the highest aims and principles of work. And since we have become able to realise better the meaning of your work, our admiration and gratitude have steadily deepened. We shall always cherish the warmest memories of our intercourse with you."

His life was not lived in vain. C. H. P. Mayo.

For 34 years (1869-1903) Levett served as chief mathematical master at King Edward's School, Birmingham. He will be remembered as one of the founders of the A.I.G.T.

He was educated at Pocklington School and St. John's College, Cambridge, taking his degree as eleventh wrangler in 1865. His life's work lay in the period when Euclid's sequence was still heavy upon schools, and in geometry his task was to do what was possible to mitigate these conditions. It is said that he wrote the scholarly *Elements of Plane Geometry*, published by the A.I.G.T. in 1884, as the work of a committee

This book is a landmark, and it may be of some interest to recall some of its features. The A.I.G.T. were not free to recommend any essential departure from Euclid's "logical sequence," for any such departure would have led to failure in university examinations. But rearrangement within the boundary of the logical sequence was permissible Euclid arranged his propositions as steps on the road to the construction of the regular polyhedra; the A.I.G.T. order aims rather at the harmonious presentation of allied groups of geometrical facts. In Euclid's scheme theorems are in a sense subordinate to problems; the object of the A.I.G.T. led naturally to a separation of problems from theorems. The whole treatment is severely logical and scholarly; for instance, there is a careful exposition of Euclid, Book V., an essential link unless a fundamentally different educational standpoint is adopted. The A.I.G.T. advocated no such fundamental change.

In the two first pages of the book there is a list of elementary constructions to be made with ruler and compass; the graduated scale is not mentioned, but the use of the protractor and scale of chords is specified in the last paragraph. I do not remember that we used ruler and compass in Levett's class room.

Perhaps the best illustration of 30 years' change of standpoint is that the first theorem in the *Elements of Plane Geometry* is "All right angles are equal to one another"; the proof occupies a page and a half.

The book was not written as a text-book, but rather as a detailed syllabus of proofs. The severity of treatment sanctioned by the A.I.G.T. in these days may suggest that geometry must have been a dull subject to learn; but it was not dull in Levett's hands. Perhaps for boys of mathematical bent the old way at its best was more effective than the new; but Levett showed us that a teacher of genius and sympathy can reconcile those historic enemies, Euclid and the average boy.

Before the beginning of this century he had given place to younger men on the committee of the M.A., but he watched their work with sympathy.

As a teacher he appeared to be equally successful with elementary and advanced classes. He was strangely popular with boys; strangely, for he was not of the conventional type of popular schoolmaster. But boys found him kindly, humorous and interesting; and they felt immediately the distinction of his nature. As they grew up their affection and esteem for Levett deepened; at gatherings of Old Edwardians the mention of his name was invariably received with enthusiasm.

During Levett's time, and afterwards, under the reign of his successor and co-author, Dr. C. Davison, Birmingham has sent a steady stream of mathematicians to Cambridge. Levett's method with a mathematical boy should be put on record. He left him alone. He had the supreme gift, in a good teacher, of knowing when not to teach. For the most part we worked away at our books; Levett would help us when we stuck; from time to time he would take us aside into some by-path he was exploring, and give us a glimpse of high and exciting things such as non-Euclidean geometry. But systematic preparation for scholarships he eschewed and derided. We imbibed from him a contempt for every kind of cram and commercialism in learning.

His older pupils were often invited to the friendly bachelor establishment at Moseley, which he shared with the brothers Hunter and Jamson Smith.

In 1903, gravely impaired health compelled him to give up his work, and he retired with Hunter Smith to live at Colwyn Bay. He survived his friend by four years. In spite of failing health and growing disabilities he kept up his interest in mathematics to the last. C. Godfrey.

GLEANINGS FAR AND NEAR.

169. For a singular instance of a young girl, "a prodigy in mathematical and musical skill," *v.* Clarendon's *Correspondence*, 1828, ii. p. 149.

170. Dr. Wollaston had a brain tumour which produced paralysis of one side of the head. But his brain remained clear, and the last moments of his life were engaged in writing some figures in arithmetical progression, to convince his friends that though his tongue was mute for ever his brain was clear.

171. Pryme asked to be placed on any Committee in which his mathematics would be of any use, so was placed on one on navigation of the Severn. It so happened that a question of pure mathematics arose. "Godson, M.P. for Kidderminster, a Wrangler of Caius, and a Scotch member, also a good mathematician were with me requested to discuss it, and the Committee agreed to abide by our decision, so impartial were they."—*Autobiog. Recollections of G. Pryme*, 1870, p. 198.

172. A *calculus*, or *science of calculation*, in the modern sense, is one which has organised processes by which passage is made, or may be made, mechanically, from one result to another. A *calculus* always contains something which it would be *possible* to do by machinery. . . .

Those who introduce *algebraical* symbols into elementary geometry, destroy the peculiar character of the latter to every student who has any mechanical associations connected with those symbols; that is, to every student who has previously used them in ordinary algebra. Geometrical reasons, and arithmetical process, have each its own office: to mix the two in elementary instruction, is injurious to the proper acquisition of both.—De Morgan's *Trig. and Double Alg.* 1849 (footnotes), p. 92.

SOME GENERAL PRINCIPLES OF ANALYTICAL GEOMETRY.

BY D. K. PICKEN, M.A.

THE strength of analytical methods is *generality*. It is not quite easy to bring out effectively the essential generality of elementary Analytical Geometry. In that connection the following points appear to be of fundamental importance:

1. (i) If P, Q, R are collinear points, the *ratio* $PR:PQ$ is *positive* or *negative*, according as the *directions* * PQ, PR are the *same* or *opposite*.

(ii) (1) *Cartesian coordinates* of the representative point P of a given plane are determined by reference to chosen axial lines xOx', yOy' in the plane, and the question of sign may be put thus, using (i): If A, B are the points at unit distance from O on the *half-lines* Ox, Oy, respectively, and if the parallels through P to the axial lines intersect the x-axis at M and the y-axis at N, the coordinates x, y of P are given by $x = OM:OA$, $y = ON:OB$. They are essentially *one-valued* functions of the position of P.

(2) With this definition of Cartesian coordinates may be correlated the definition of *Trigonometric Functions* of an angle, thus:

If the Cartesian frame is *rectangular*—more precisely, if $\angle AOB$ (or $\angle xOy$) is a *positive* right angle—and if *P is on the unit-circle* (through A and B) of centre O, we have the standard diagram of Trigonometry, and

$$\cos AOP = OM:OA, \quad \sin AOP = ON:OB;$$

i.e. the cosine and sine of an angle are rectangular Cartesian coordinates of the point P determined by the angle, on the unit-circle, in the standard diagram of Trigonometry. (The best *general definition* of $\tan\alpha$ is $\sin\alpha/\cos\alpha$; etc.)

The specification of direction * by Cartesian coordinates of a point at unit distance from the origin is, of course, the *general* Cartesian principle—for *any* Cartesian frame (in space).

(iii) (1) The *Polar coordinates* r, θ of P, with reference to Ox, may be defined thus: θ is (the circular measure of) *any* angle that *either direction* * of the line through O and P makes with Ox; and $r = OP:OU$, if U be the point at unit distance from *O in the θ-direction*. Thus r may have one or other of two opposite values, and θ has a corresponding *double* infinity of values.

This many-valuedness of Polar coordinates makes them fundamentally less simple than Cartesian coordinates, but appears, on the other hand, to be essential to their most effective theoretical use (see below).

(2) Polar coordinates are correlated with Rectangular Cartesian coordinates, based (trigonometrically) on Ox, as follows:

$$x = OM:OA = (OM:OH)\times(OH:OA) = (OP:OU)\times(OH:OA) = r\,.\cos\theta$$

if H is related to U as M to P; and, similarly,

$$y = r\,.\sin\theta.$$

(3) In the equivalent form $x/\cos\theta = y/\sin\theta = r$, these *general* relations give the Cartesian equation of the general straight line through the origin—determined by *given* θ-direction.*

2. The "change-of-origin" transformation $x = x_0 + \xi$, $y = y_0 + \eta$—simplest and most powerful of general "methods"—gives the *general* equation of The Straight Line, through point (x_0, y_0) and having ψ-direction,* viz.:

$$(x - x_0)/\cos\psi = (y - y_0)/\sin\psi = \rho,$$

* It is to be understood that a straight line has two opposite directions—not one "direction" with two opposite "senses," as it used to be expressed.

in which ρ, ψ are Polar coordinates of P with reference to $P_0\xi$, so that ρ is positive or negative according as the direction * P_0P is or is not the ψ-direction.*

The parametric equivalent $x=x_0+\rho \,.\, \cos\psi$, $y=y_0+\rho \,.\, \sin\psi$ exhibits the correlation with Polar coordinates still more obviously.

3. (i) For the "Perpendicular Form" $\cos\alpha \,.\, x+\sin\alpha \,.\, y=p$ of the Straight Line equation, the quantities p, α may be specified as Polar coordinates of the point L, such that OL is the perpendicular from O to the line—Polar coordinates in the general sense of § 1, (iii), (1). The equation is equivalent to $r \,.\, \cos(\theta-\alpha)=p$, and, as such, is correlated for *generality* with $r \,.\, \cos\theta=x$, with change to α-direction * as direction of reference.

(ii) The fact that α may belong to *either* perpendicular direction,* and that p may be of either sign, makes *general* determination of the perpendicular distance from a point P' quite simple. Using the change-of-origin transformation, to P', viz. $x=x'+\xi$, $y=y'+\eta$, we get the $\xi\eta$-equation

$$\cos\alpha \,.\, \xi+\sin\alpha \,.\, \eta=p-\cos\alpha \,.\, x'-\sin\alpha \,.\, \eta'=\varpi, \text{ say.}$$

The P' frame of reference being *directionally* * the same as the original frame, this equation is also of the "Perpendicular Form," as defined above. Hence ϖ, α are Polar coordinates, with reference to $P'\xi$, of the point K such that $P'K$ is the perpendicular from P' to the line; and ϖ is positive or negative according as the direction * $P'K$ is the α-direction or the opposite.

The Cartesian coordinates of K are given by

$$x=x'+\xi=x'+\varpi \,.\, \cos\alpha, \quad y=y'+\eta=y'+\varpi \,.\, \sin\alpha.$$

(iii) The reduction of the general equation of the first degree, viz.

$$a \,.\, x+b \,.\, y+c=0,$$

to the Perpendicular Form requires

$$\cos\alpha/a=\sin\alpha/b=1/(\pm\sqrt{a^2+b^2})=-p/c,$$

the ambiguity of sign corresponding to the choice between two opposite directions.*

And $\quad \varpi=p-\cos\alpha \,.\, x'-\sin\alpha \,.\, y'=(a \,.\, x'+b \,.\, y'+c)/(\mp\sqrt{a^2+b^2}).$

The coordinates of K are given by

$$x=x'+\varpi \,.\, \cos\alpha=x'-a \,.\, (a \,.\, x'+b \,.\, y'+c)/(a^2+b^2), \text{ etc.,}$$

from which, of course, the ambiguity eliminates itself.

4. Those who, like the writer, have striven for many years to get a convincing presentation of the generality of the important facts of §§ 2 and 3 will recognise the effectiveness of correlating these facts with the definition of Polar coordinates, as given in § 1. I would add that, so far as my experience goes, it is never, in the long run, wise to make *unnecessary* restrictions, like the restriction of r (and, commonly, of p) to *positive* values. (The specification of one value as "the principal value" seems to be the right kind of procedure.)

It is interesting to note that the relation of these facts to the corresponding facts of Analytical Solid Geometry—replacing the angle-specification of direction * by (l, m, n)—makes itself apparent, without any straining after that effect.

Ormond College, University of Melbourne. D. K. PICKEN.

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173. Bossuet n'a laissé apercevoir dans aucun temps de sa vie du goût pour l'étude des mathématiques.—Bausset's *Histoire de Bossuet*, i. 16, Paris, 1814.

De toutes les sciences, celle des mathématiques fut la seule dont Bossuet ne donna pas lui-même des leçons (to the dauphin, his pupil).—*Loc. cit.* p. 367.

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* See previous footnote.

EXTENSION OF SOME DEFINITIONS AND PROPOSITIONS IN EUCLID'S BOOK XI. AND REMARKS.

By R. F. Muirhead, M.A.

Euclid does not define the inclination to one another, or the mutual perpendicularity of non-intersecting straight lines. (For brevity I shall omit the adjective *straight* before *lines.*)

The following definition is justified by the 10th Proposition of Bk. XI. Def. : *The inclination of two non-intersecting lines to one another, or the " angle between them " is the angle between two intersecting straight lines which are parallel to them respectively.*

(Remark that the same proposition is required to justify Euclid's definition of the angle between two planes. In both cases there is a certain ambiguity as between one angle and its supplement. In the case of *lines* it is removed if the *senses* of the lines be specified.)

Definition of mutual perpendicularity of two non-intersecting lines : *They are mutually prependicular if the angle between them is a right angle.*

The extension of XI. Prop. 4 is as follows :

If a straight line is perpendicular to two non-parallel straight lines in a plane, it is perpendicular to every straight line in the plane. Here the restriction to lines which the perpendicular meets is removed. The proof is obvious.

The extension of XI. Prop. 5 (which is a converse of Prop. 4) is as follows :

If any number of lines $l_1, l_2, \dots l_n$ which intersect one another so as to form a connected figure are perpendicular to the same line L, then they lie in the same plane perpendicular to L.

The proof of this requires a *Lemma*, which is the correlative of Prop. 13. *Through a given point one and only one plane can pass which is perpendicular to a given line.* The proof of this is obvious.

The proof of the extension of XI. 5 can then be stated thus : If l_1 and l_2 intersect one another and are perpendicular to L, then they lie in a plane P perpendicular to L. Similarly, if l_2 and l_3 intersect each other and are perpendicular to L, they lie in a plane perpendicular to L which, by the Lemma, must coincide with P. Again, if l_4 intersect l_1 or l_2 or l_3, it must by similar reasoning lie in plane P. Continuing this reasoning, we prove that all the lines $l_1, l_2, \dots l_n$ lie in the same plane P perpendicular to L, provided they form a *connected* figure.

These extensions simplify notably the proofs of many theorems in Solid Geometry. I give two examples :

I. The " Theorem of the Three Perpendiculars."

A' is the projection on a plane P of a point A without it and $A'B$ is the perpendicular from A' on a line CD in plane P. Prove that AB is perpendicular to CD.

Proof. AA' is $\perp$ to $A'B$ and to CD.

$\therefore$ CD is $\perp$ to AA' and to $A'B$; $\therefore$ to plane $AA'B$; $\therefore$ to AB.

II. OA, OB, OC are three mutually perpendicular lines and ON is the perpendicular from O on the plane ABC. It is required to prove that N is the orthocentre of the triangle ABC.

By extension of XI. 4 ON is $\perp$ to AB and OC to AB ; $\therefore$ by extension of XI. 5 AB is $\perp$ to plane NOC ; $\therefore$ to NC. Similarly NA is $\perp$ to BC and NB to CA. Hence N is the orthocentre.

Cor. If CN produced meets AB in F, AB is $\perp$ to plane $COFN$; $\therefore$ to OF.

Converse No. 1. If N is the orthocentre of $\triangle ABC$, OA, OB, and OC being mutually perpendicular, then ON is $\perp$ to plane ABC.

For AB is $\perp$ to CO and to CN; $\therefore$ to plane CON; $\therefore$ to ON. Similarly BC is $\perp$ to ON; $\therefore$ ON is $\perp$ to plane ABC.

Converse No. 2. With $OABC$ as before, if OF is $\perp$ to AB and $ON \perp$ to CF, then N is the orthocentre of triangle ABC.

For OF and OC are both $\perp$ to AB; $\therefore$ AB is $\perp$ to plane COF; $\therefore$ AB is $\perp$ to ON. Thus ON is $\perp$ to CF and to AB; $\therefore$ to plane ABC.

Hence, as in the original proposition, N is the orthocentre of triangle ABC.

Remark : Euclid's definition of the perpendicularity of one *plane* to another is unsymmetrical, and leaves open the question whether plane Q is $\perp$ to plane P if plane P is $\perp$ to plane Q according to definition. Of course it is easy to prove the affirmative. That having been done, we could speak of planes mutually perpendicular. R. F. MUIRHEAD.

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174. Counting or numeration . . . proceeds from 0 which represents, and must represent, the state of mind with respect to the number attained, before the counting begins.—De Morgan's *Trig. and Double Alg.* 1849, p. 115.

An algebra similar to ours requires a starting symbol, 0, wholly ineffective in its own operation, so that $0+0=0$, $0+A=A$.—De Morgan's *Trig. and Double Alg.* 1849, p. 166.

175. God Geometrises. Diderot refers to "the ingenious expression of an English geometer (*Oeuvres*, xix. 294, Assezat & Tourneux, 20 vols.) that "God geometrises." "He is unaware apparently of the tradition which attributes the expression to Plato, though it is not found in Plato's writings. Plutarch, I believe, is the first person who mentions the saying, and discusses what Plato exactly meant by it. In truth it is one of that large class of dicta which look more ingenious than they are true. There is a fine Latin passage by Barrow on the mighty geometry of the universe, and the reader of the *Religio Medici* may remember that Sir Thomas Browne pronounces God to be like " a skilful geometrician."—Morley's *Diderot*, i. 104, 1891.

The passage containing the attribution of ὁ θεὸς γεωμετρεῖ to Plato is in Plutarch's *Sympos.* viii. 2. God is often depicted in mediaeval literature as marking out a circle with compasses—*e.g.* v. *Yorkshire Archaeological Journal*, iv. 319, 375. In a stained-glass window in St. Edmund's Church, Salisbury, was seen " God the Father creating the Sun and Moon with a pair of compasses in His hand, as if He had done it according to some Geometrical Rules," and the same notion is found in the frontispiece to Comber's *Companion of the Temple*, 1684. The window so offended the sense of propriety of Henry Sherfield, the Recorder of the city, that he incontinently broke the window with his stick, and was condemned by the Star Chamber, 1672. In Erasmus, his Paraphrase of St. Mark, we find: " by a carpenter mankind was created and made, and by a carpenter mete it was that man should be repaired." The passage from the *Religio Medici* is in c. I., sect. xvi. v. (Greenhill), or p. 23, *Willis Bund*, 1879: For God is like a skilful geometrician, who, when more easily, and with one stroke of his compass, he might describe or divide a right line, had yet rather do this in a circle or longer way, according to the constituted and forelaid principles of his art. . . . And Milton pictures the " omnific Word " as

in His hand
He took the golden compasses, prepar'd
In God's eternal store, to circumscribe
This universe. . . . *Paradise Lost*, vii. 225.

Barrow prefixes Plutarch's phrase to a Latin prayer beginning: Tu autem, Domine, quantus es Geometra ! adding, " God acts the part of a geometrician . . . His government of the world is no less Mathematically exact than his Creation."

THE SYMBOL FOR ZERO.

THE following letter and the reply explain themselves :

TELEGRAMS
"ELECTRICITY WORKS, NORTH SHIELDS"
TEL. NO. 427
C. TURNBULL, M.I.E.E.
BOROUGH ELECTRICAL ENGINEER.

REF. { *Yours* / *Ours* CT/IS.

COUNTY BOROUGH OF TYNEMOUTH,
ELECTRICITY WORKS,
NORTH SHIELDS,
Feb. 1923.

SIR,

Much of the difficulty that learners find, when they begin higher mathematics, is due to the fact that the symbol "0" has two entirely different meanings, and it is not made clear to them which meaning is involved in the individual functions which they are trying to understand. In his early work the student's idea of "0" is expressed by the equation $x-x=0$. When he comes to higher mathematics, sometimes he is dealing with this 0 and sometimes with the other 0, which is the value of $\frac{x}{\infty}$, x being finite. It would clear the air if the second 0 was expressed by a different symbol, say $\bar{0}$.

The learner would then be able to understand the different properties of the two nothings : such as that while $3\times 0=4\times 0$, $3\times\bar{0}$ does not necessarily equal $4\times\bar{0}$. Also $\frac{x}{0}$ has no meaning, but $\frac{x}{\bar{0}}=\infty$, x being finite. The meaning of the equation $\sin\bar{0}=\bar{0}$ becomes quite clear. The equation given in so many text-books that $\sin\theta=\theta$ when $\theta=0$ has worried numberless students, and it is of course incorrect.

The equation $\frac{\theta-\sin\theta}{\tan\theta-\theta}=\frac{1}{2}$ when $\theta=0$ has a mysterious appearance, where the text-book has stated that when $\theta=0$, $\sin\theta$, θ, and $\tan\theta$ are all identical, but it becomes simple when the symbol $\bar{0}$ is used.

It would also become clear that $\log 0$ has no meaning, as it stands for $\log(x-x)$; on the other hand $\log\bar{0}$ is equivalent to $\log x-\log\infty$, or to $\log\frac{x}{\infty}$, x being finite, and therefore it has a meaning.

The statements in text-books that 1^∞ and 0^∞ have values other than 1 and 0 are very puzzling, and actually incorrect. On the other hand $(1+\bar{0})^\infty$ and $\bar{0}^\infty$ have values other than 1 and 0, quite evidently.

To the trained mathematician it may appear that students ought to understand these things, but from actual experience I can affirm that many men, possessed of sound engineering instincts, are quite unable to comprehend mathematics as presented by many standard text-books, but they do see the meaning when presented with the symbol $\bar{0}$.—Yours faithfully,

C. TURNBULL.

ROOM 9, UNIVERSITY COLLEGE,
BANGOR, 19*th Feb.*, 1923.

MY DEAR SIR,

Your letter greatly interested me, because I feel that your proposals for the different meanings of the symbol for zero arose from difficulties very

similar to those I have been endeavouring to meet in a totally different way as very roughly indicated in my letter to *Nature*.

I enclose a few "stencils" which I have had duplicated for the use of my students learning Calculus, and I think you will see that while your plan is to have separate notations for vanishing quantities, my policy is to avoid the use of zero altogether till the students have got familiar with the rudiments of the Calculus. On glancing at the stencils before sending them to you, I find that this plan has actually been carried out in them. The rule I tell my students is that they may always put two things equal to each other, provided they do not bring any zeros into the equations, but they must be very careful not to put anything equal to zero until they know the rules of the game. You will see that I start with $(y_2 - y_1)/(x_2 - x_1)$ and reduce it to a form in which they can make x_1 and x_2 *equal* without making the function assume the form 0/0.

Now all the text-books adopt the opposite practice, and seem to try to throw as many stones of the form 0 as possible at our unfortunate students' heads. They cannot even let the student make b equal to a or even talk of such a well-known thing as b minus a, but they must first put b equal to a plus delta a and then put delta a equal to 0. Moreover, there are some things you may put equal to 0 and some things you mayn't, and the former are mysteriously described as quantities of the second order.

Now the second order intruders as a rule have no business to come into the formulae in the way they usually are introduced, and they are not adequately got rid of by the conventional plan of merely neglecting them. The fault arises from the fact that the writers always represent $\Delta f(x)$ by $f(x+\Delta x) - f(x)$, although it would be equally logical to take $\Delta f(x) = f(x) - f(x - \Delta x)$, and as a matter of fact the latter plan is often used in practice. We do not estimate the rise or fall of the German exchange to-day by the amount that the value will have increased or decreased by to-morrow, which we do not know, but by the increase or decrease over the value of yesterday.

Similarly, for annual increases of population, rises or falls of stocks and shares, etc., the formula always is

$$\Delta \text{ value} = \text{present value} - \text{previous value},$$

which is of the form

$$\Delta f(x) = f(x) - f(x - \Delta x).$$

My contention then is that an introduction to the notion of a differential coefficient based on the first of the two forms is inadequate and misleading, and that unless the more general form is adopted, which is outlined in my stencils, the values of $\Delta f(x)$ should be studied for intervals Δx both before and after the value x of the variable.

Taking the simplest case $f(x) = x^2$, we have

$$(x+\Delta x)^2 - x^2 = 2x\,\Delta x + \Delta x^2;$$

but

$$x^2 - (x - \Delta x)^2 = 2x\,\Delta x - \Delta x^2$$

(which on dividing by Δx give respectively $2x + \Delta x$ and $2x - \Delta x$).

Now the presence of the term Δx^2 is entirely due to the fact that we have considered only a one-sided variation Δx. It is an error of judgment due to the fact that in the first case we have found only the increase of value of x^2 over a range Δx of values of x all greater than the original x, and the further that range extends beyond x the greater the deviation. In the other case the error of judgment reduces the result by Δx^2. Considering the quotients, the only value which always lies between the two estimates, however small, is $2x$, which therefore properly measures the rate of change of x^2 in the immediate vicinity of a value x. But you will see that the so-called terms of higher order which are so lavishly introduced into our books have no actual significance whatever, and no definite value, since they only depend on the one-sidedness of the variation Δx employed in estimating the rate of increase of

the function, and they are different for the two sides of the value to which the variable approximates.

We have in fact an instance in which it may be said that there are two sides to a question, and most people only see one.

The proposal contained in your letter appears to me to be a good one, but why not use the letter *o*, capital or small, but preferably small, as being more distinct from the numeral ? Very likely I shall introduce this idea in my classes in connection with virtual work and such things.

This is not, however, the only case in which 0 has an ambiguous meaning, for in figures such as 96,000,000 miles or £400,000,000, the cyphers do not mean that there are no units, tens, hundreds, thousands, etc., in the amount, but only represent blanks. This matter was referred to by me some years ago in a letter to the *Mathematical Gazette*, but hardly anyone took any notice of it. In at least one text-book, however, the distinction has been noticed, and the writer has used the symbol × for the unknown digits. This was rather a pity, as it is the multiplication sign. I am now of opinion that the best symbol to use is the letter *b*, standing for " blank," which occupies the same space in printing as a numeral, thus 96,*bbb*,*bbb*.

I shall be very glad if you will kindly send your letter with this reply from me to the *Mathematical Gazette*, and if you care to enclose one set of my stencils,* I shall be quite willing that the Editor should publish these also should he see fit.—Yours faithfully, G. H. BRYAN.

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176. Alchemists used to be called " multipliers."—Disraeli's *Curiosities of Literature*, p. 103.

177. In 1872 a writer in the *Edinburgh Review* remarks with apparent approval: We have heard of a professor of mathematics who used to say that in his first session he was only one problem ahead of his students. But one problem was enough. He was never overtaken.

178. Moukhtar Pasha afterwards gave me what I believe to be a very interesting book on Astronomy and Mathematics, written by himself, when he was tutor to Prince Izzedin, son of the Sultan Abdul-Aziz. It was a dissertation on the early Oriental systems, and on the principles whereby mathematical instruments were constructed. I sent a copy to Professor Jowett, then Vice-Chancellor of Oxford, wishing if possible to obtain the opinion of some English mathematician on the subject. The Professor answered, however, that this was impossible, for those at Oxford who knew Turkish did not know mathematics, and none of the mathematicians knew any Turkish.—*Rambling Recollections*, The Right Hon. Sir Henry Drummond Wolff (1908), ii. pp. 300-301.

179. Charles James Fox, at Hertford College, 1764, liked mathematics " vastly," and said: " I believe they are useful, and I am sure they are entertaining."

Trevelyan says: " Pursuing them with zest at the age when they most rapidly and effectively fulfil their special function of bracing the reasoning faculties for future, he got more profit from them than if he had been senior wrangler."—*Early Life of Charles James Fox*, p. 58.

As to Trigonometry, it is a matter of entire indifference to the other members of the College whether they proceed to the other branches of mathematics immediately, or wait a term or two longer. You need not therefore interrupt your amusements by severe studies, for it is wholly unnecessary to take a step onwards without you, and therefore we shall stop until we have the pleasure of your company.—From Dr. Newcome, Principal of Hertford, to Fox in Paris or London.

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* The stencils will appear later.

MATHEMATICAL NOTES.

670. [O. 2. c.] *Fagnano's Theorem on Arcs of an Ellipse.*

The following is the result of an attempt to deduce a proof of the above theorem from the geometry of the article on " Elliptic Trammels and Fagnano Points " in Nos. 92 and 93 of the *Mathematical Gazette.* It has been written after reading Mr. Langley's note in No. 149, December 1920.

In Fig. 5, etc., of the above article, the point P is made to describe the ellipse, of which BPA is the first quadrant, by the movement of the trammel line HK between rectangular axes. It is shown (Art. 6) that ϕ, the eccentric angle of P, is equal to the angle CHK ; and hence if HK slides so as to make P move along the ellipse towards B, the value of ϕ increases, or $d\phi$ is positive. But if P is made to move towards A, then $d\phi$ is negative. The instantaneous motion of P is one of rotation about O (Fig. 5).

Hence $$d(AP)=OP\,d\phi, \quad d(BP)=-OP\,d\phi.$$

Further, combining Figs. 5 and 7, we have $CD=CZ=OP$, where CD is the semi-diameter conjugate to CP.

Hence $$d(AP)=CD\,d\phi \quad \text{and} \quad d(BP)=-CD\,d\phi.$$

Next, referring to P_1 and P_2, a Fagnano pair of points (Fig. 15) and to the corresponding quadrilateral (Figs. 14 or 16), it is shown (Art. 26) that if ϕ_1 and ϕ_2 are the eccentric angles of P_1 and P_2, then $\tan\phi_1 \tan\phi_2=\frac{b}{a}$.

If P_1 and P_2 move along the ellipse towards one another, remaining a Fagnano pair, and if BP_1 and AP_2 are denoted by σ_1 and σ_2, then

$$d\sigma_1=-CD_1 d\phi_1 \quad \text{and} \quad d\sigma_2=CD_2 d\phi_2,$$

where CD_1 and CD_2 are the semi-diameters conjugate to CP_1 and CP_2.

As a second way of measuring $d\sigma_1$, we may note that the tangent at P_1 is parallel to the trammel line through P_2 (Art. 28, Fig. 18), and that the radius of curvature at P_1 is $\frac{CD_1^2}{CD_2}$ (Art. 35), and hence $d\sigma_1=\frac{CD_1^2}{CD_2}d\phi_2$.

Equating the two values of $d\sigma_1$, we find that

$$\frac{-d\phi_1}{CD_1}=\frac{d\phi_2}{CD_2} \quad \text{and} \quad \text{each}=\frac{-d(\phi_1-\phi_2)}{CD_1+CD_2}.$$

Transferring now to the corresponding Fagnano quadrilateral, Fig. 16, and noting in that figure that the angles $c_1kz=\frac{\pi}{2}-\phi_1$ and $kc_1z=\phi_2$, we see that, if the angle c_1zh is denoted by ω, then $\omega=\frac{\pi}{2}-\phi_1+\phi_2$ and $d\omega=-d(\phi_1-\phi)$.

Hence $$d\sigma_1=-c_1z\,d\phi_1=\frac{c_1z^2d\omega}{c_1z+c_2z} \quad \text{and} \quad d\sigma_2=\frac{c_2z^2d\omega}{c_1z+c_2z};$$

$$\therefore\ d\sigma_1-d\sigma_2=(c_1z-c_2z)d\omega.$$

But $$c_2z=c_1m \text{ (Art. 24)};$$

$$\therefore\ d\sigma_1-d\sigma_2=mz\,.\,d\omega.$$

Now, in Fig. 14, if m moves through a small arc of the dotted circle toward z, it is easily seen that

$$mz\,.\,d\omega=d(pm),$$

and therefore $$d\sigma_1-d\sigma_2=d(pm),$$

where pm is the common Fagnano distance of P_1 and P_2.

Hence, since σ_1, σ_2 and pm vanish together,

$$\sigma_1-\sigma_2=\text{the Fagnano distance},$$

which is Fagnano's theorem.

Note.—I have a few copies of the paper on "Elliptic Trammels and Fagnano Points" still by me, and will send them (as far as is possible) to members of the Association who would like to have a copy, but are unable to refer to Nos. 92 and 93 of the *Gazette*. P. J. HARDING.

671. [X. 20.] *The Solitary Seven.*

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            × × ×
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Solution (unique). Divisor = 124. Quotient = 97809.

[This "missing figure" problem appeared, with its solution, in the *Strand Magazine* for Nov. and Dec. 1922. We have to thank the author, the Rev. Egbert F. Odling, of St. Paul's Vicarage, Huddersfield, for leave to reprint it here. Nor must we omit our acknowledgments to the *Strand* for their ready permission, and to Mr. Henry Ernest Dudeney who placed us in communication with the author. Mr. Dudeney points out that "the 7 is indeed solitary, since it appears nowhere in the working of the solution, except in the one position given."—EDITOR.]

672. [J. 2. c.] "*Put and Take.*"

The expectation of gain to the next spinner when there are n counters in the pool $=\dfrac{n}{22}-\dfrac{1}{121}-0{\cdot}190516(-0{\cdot}4252399)^n\,.\,\sin(n\,.\,79°\ 15{\cdot}575'-2°\ 29{\cdot}175')$

$=0{\cdot}11606$ with 1 or 3 counters.
0·06865 „ 2 „
0·17799 „ 4 „
0·22047 „ 5 „
0·26343 „ 6 „

W. HOPE-JONES.

673. [K¹. 21. a. α.] *To find* $\sqrt{11}$.

Draw AB 11 units long.

In AB place point C as near the middle as possible to avoid fractions. ($AC=5$ and $BC=6$.)

Through C draw perpendicular CO.

With A as centre, and CB as radius, draw an arc cutting CO in O.

$CO=\sqrt{11}$.

Folkton Rectory. E. J. F. CANHAM.

674. [A. 3. g.] *Note on a numerical method of solving the equation* $x^n = ax + b$.

(Illustrations given refer to $x^3 = ax + b$, but the process is general.)

There are two ways of going to work which I here call (i) *The Cube Method*, (ii) *The Cube Root Method*. Both are very expeditious and need nothing but a table of cubes and cube roots.

(i) *The Cube Method.* Assume any value of x, find x^3, equate to $ax + b$ and solve for x. Cube this again and repeat. The values of x soon approach a definite quantity, and the same figures begin to recur, when the approx. root is found.

(ii) *The Cube Root Method.* Assume any value of x, find $ax + b$ and equate to x^3. Take the cube root for the next value of x, and repeat.

Mistakes in computation are of no moment. They only delay the result or may even hasten it. As an illustration I give the working to find the roots near -2 and 5 of $x^3 - 20x - 30 = 0$.

Root near -2. *Cube Method.*

x	-2	$-1{\cdot}9$	$-1{\cdot}843$	$-1{\cdot}813$	$-1{\cdot}798$	$-1{\cdot}790$	$-1{\cdot}786$	$-1{\cdot}783$
x^3	-8	$-6{\cdot}86$	$-6{\cdot}26$	$-5{\cdot}96$	$-5{\cdot}81$	$-5{\cdot}73$	$-5{\cdot}69$	
$x^3 - 30\ (=20x)$	-38	$-36{\cdot}86$	$-36{\cdot}26$	$-35{\cdot}96$	$-35{\cdot}81$	$-35{\cdot}73$	$-35{\cdot}69$	

and the root is not far from $-1{\cdot}783$.

Root near 5. *Cube Root Method.*

x	5	5·07	5·084	5·088	5·089
$20x$	100	101·4	101·68	101·76	101·78
$20x + 30\ (=x^3)$	130	131·4	131·7	131·8	

and the root is very near 5·089.

This method is not new (I cannot quite remember where I got it), but on trying it recently, it seemed to have some curious vagaries. Instead of approximating to the required root it sometimes wanders away and converges to a different one, and sometimes diverges altogether. Thus interchange the methods above, and the cube method beginning with $x = 5$ leads to the root $-1{\cdot}783$, and the cube root method beginning with $x = -2$ leads to the third root near -3; while the cube root method always diverges for the equation $x^3 + 20x - 30 = 0$ and never gives a root at all. The object here is to discuss these peculiarities and find a working rule.

Draw the graphs of $y_1 = x^3$ and $y_2 = ax + b$.

There are two main cases.

Case I. a positive. (The sign of b is immaterial.) The figure shows the

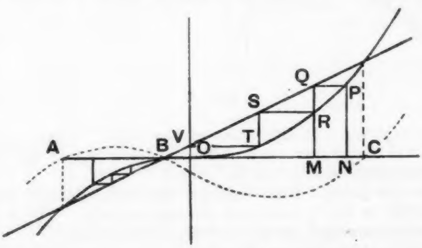


FIG. 1.

case of three real roots, but the discussion is the same when there is only one. The dotted curve is $y = x^3 - ax - b$.

(i) *Cube Method.* Take any value ON of x, find x^3, *i.e.* NP or y_1, and equate to $y_2\ (=QM)$, solve for $x\ (=OM)$, and repeat. Fig. 1 shows the next few steps graphically, by drawing $RSTV\ldots$, and we are obviously approaching the root OB. The same holds if we begin anywhere between A and B (also Fig. 1), but x rapidly diverges if we start left of A (Fig. 2), or right of C (Fig. 3).

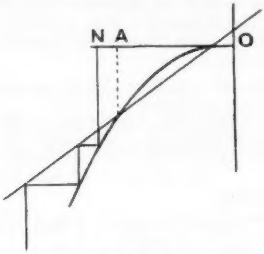


FIG. 2.

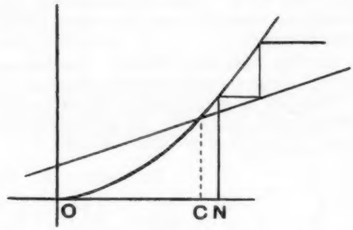


FIG. 3.

(ii) *Cube Root Method.* Take any value ON of x, find $ax+b\ (=PN$ or $y_2)$, equate to $x^3\ (=QM$ or $y_1)$, and take cube root for $x\ (=OM)$ and repeat. Fig. 4 illustrates the next few steps again, and shows that, beginning anywhere to the right of B, we approximate to the root OC, and anywhere to the left of it, to the root OA.

The distinction obviously depends on the direction of slope of the dotted curve. (See practical rule given at the end.)

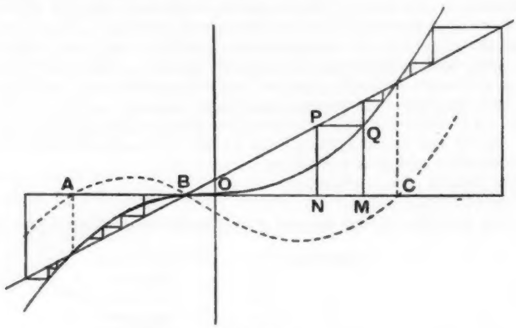


FIG. 4.

Case II. a negative. (b again has either sign.) There is now of course only one root, and the proper method depends on whether the curve or line has the greater slope numerically, *i.e.* $3x^2 \gtrless |a|$.

First suppose $3x^2 > |a|$. Fig. 5 shows the cube method beginning left or right of A ($x=ON$ or ON'), and both diverge rapidly. Fig. 6 shows that the cube root process converges to the root from either side.

It will be seen that the values of x obtained are alternately above and below the root, and after finding two it saves time in practice to jump to a value between them.

Next, when $3x^2 < |a|$. Similar figures, not here given, show that the cube

root process diverges, the cube process converges to the root, beginning either side of it.

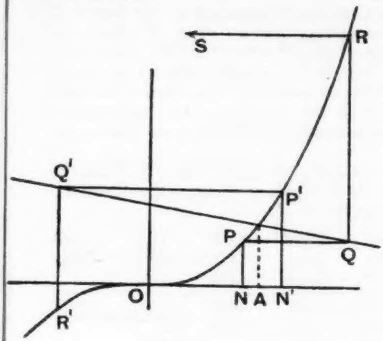


FIG. 5.

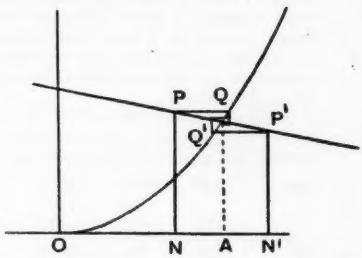


FIG. 6.

If $3x^2=|a|$ *very nearly.* Fig. 7 shows that the convergence (or divergence) is very slow, but the root is obviously about half way between two consecutive values of x, which suggests a method of shortening the work.

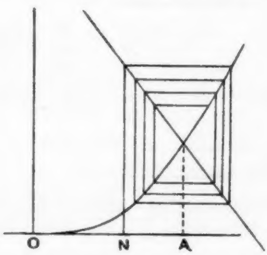


FIG. 7.

Thus, to find the root near 2 of $x^3+12x=31$. Cube root process.

x	2	1·91	2·06	x diverges. Try $x=1{\cdot}955$ by cube process.
$12x$	24	22·92		
$31-12x=x^3$	7	8·08		

x	1·955	1·961	1·955	Try 1·958	1·958
x^3	7·472	7·541		7·506	
$31-x^3=12x$	23·528	23·459		23·494	

and root $=1{\cdot}958$.

It now appears that the practical rule is:

To solve $x^3 \pm ax+b=0$, test the sign of $3x^2-a$ in the neighbourhood of a root, and use the $\left.\begin{matrix}\text{cube root}\\ \text{cube}\end{matrix}\right\}$ process according as the result is $\left.\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$.

Similar results hold for equations of higher degree, with certain modifications that can easily be tested as above. For example, both roots of $x^4+3x-20=0$ can be found, starting from any number between them by using the two real fourth roots, but the fourth power method fails to give either.

J. D. M'NEILE.

675. [A. 5. a.] *Partial Fractions associated with Quadratic Factors.*

In illustration of the article on this subject in the *Gazette*, No. 156, p. 10, I give the extraction of the elements corresponding to the factor Y of the function N/Y^2Z^2, where, in descending powers of x, $Y=2+1-2$, $Z=1-1+3$, $N=24-102-95+296-90-147+79-70+23$. To leave the work typical, I have refrained from accidental simplifications; in particular, powers of 2 have been allowed to accumulate. The sign $\sim$ is used to denote equivalence for the purpose of the problem; two expressions connected by this sign are in fact asymptotically equivalent near both roots of the function Y.

Since the second power of Y is to be retained, we must multiply N and Z^2 by the fourth power of the coefficient of x^2 in Y in order to be sure that fractions are not introduced in the preliminary divisions by Y.

384	-1632	-1520	+4736	-1440	-2352	+1264	-1120	+ 368
		384	-1824	- 224	3024	-3176	2260	-3042
	- 192	912	112	-1512	1588	-1130	1521	
384	-1824	- 224	+3024	-3176	+2260	-3042	+2661	-2674
		384	-2016	1168	424	-2220		
	- 192	1008	- 584	- 212	1110			
384	-2016	+1168	+ 424	-2220	+3794	-5262		
		384	-2208	2656				
	- 192	1104	-1328					
384	-2208	+2656	-3112	+ 436				

16	-32	+112	- 96	+144
		16	- 40	148
	- 8	20	- 74	
16	-40	+148	-210	+292
		16		
	- 8			
16	-48	+164		

Thus $$\frac{N}{Z^2}\sim\frac{(2661-2674)+\tfrac{1}{2}(3794-5262)Y+\tfrac{1}{4}(-3112+436)Y^2}{(-210+292)+\tfrac{1}{2}(-48+164)Y+\tfrac{1}{4}.16Y^2}.$$

To complete the square in Y, using whole numbers only, we have to take $8Y=16+8-16=y^2-17$ with $y=4x+1$. Then

$$\frac{N}{Z^2}\sim\frac{(2661+1897Y-778Y^2)(y-1)+(-10696-10524Y+436Y^2)}{(-210-24Y)(y-1)+(1168+328Y+16Y^2)}$$

$$\sim\frac{(2661+1897Y-778Y^2)y+(-13357-12421Y+1214Y^2)}{(-210-24Y)y+(1378+352Y+16Y^2)}.$$

Multiplying numerator and denominator by

$$(210+24Y)y+(1378+352Y+16Y^2),$$

and replacing y^2 by $17+8Y$, we have

$$\frac{N}{Z^2}\sim\frac{\begin{array}{l}\{(2661+1897Y-778Y^2)(1378+352Y+16Y^2)\\ \qquad+(-13357-12421Y+1214Y^2)(210+24Y)\}y\\ +\{(2661+1897Y-778Y^2)(210+24Y)(17+8Y)\\ \qquad+(-13357-12421Y+1214Y^2)(1378+352Y+16Y^2)\}\end{array}}{-(210+24Y)^2(17+8Y)+(1378+352Y+16Y^2)^2};$$

whence merely by expanding each product as far as the term in Y^2, an arith-

metical operation which, if a little tedious, is trivial compared with the development even of a small determinant,

$$\frac{N}{Z^2} \sim \frac{\begin{array}{l}(861888+621760Y-404928Y^2)(4x+1)\\ \qquad +(-8906176-9489344Y-1218624Y^2)\end{array}}{1149184+445952Y+77568Y^2}$$

$$\sim \frac{\begin{array}{l}(3447552+2487040Y-1619712Y^2)x\\ \qquad +(-8044288-8867584Y-1623552Y^2)\end{array}}{1149184+445952Y+77568Y^2}.$$

$$\begin{array}{rrr} 3447552+2487040 & -1619712 \\ & -\ \ 232704 \\ -1337856 & -\ \ 445952 \\ \hline 3447552+1149184 & -2298368 \end{array} \qquad \begin{array}{rrr} -8044288 & -8867584 & -1623552 \\ & & 542976 \\ & 3121664 & 2229760 \\ \hline -8044288 & -5745920 & +1149184 \end{array}$$

Thus ultimately

$$N/Z^2 \sim (3+Y-2Y^2)x+(-7-5Y+Y^2),$$

and therefore
$$\frac{N}{Y^3Z^2} \sim \frac{3x-7}{Y^3}+\frac{x-5}{Y^2}-\frac{2x-1}{Y}.$$

If we wish to find the remainder when the elements associated with Y are subtracted from N/Y^3Z^2, we have only to remember that the process of synthetic division provides a quotient as well as a remainder. Identically

$$\begin{aligned}16N=(2661-2674)+(1897-2631)Y\\ +(-778+109)Y^2+\tfrac{1}{8}(384-2208+2656)Y^3,\end{aligned}$$

and therefore

$$\begin{aligned}
&16N-16Z^2\{(3+Y-2Y^2)x+(-7-5Y+Y^2)\}\\
&\quad=\{48Y^3-(-210-24Y)(3+Y-2Y^2)\}x^2\\
&\quad+\{(2261+1897Y-778Y^2-276Y^3)-(-210-24Y)(-7-5Y+Y^2)\\
&\qquad\qquad -(292+82Y+4Y^2)(3+Y-2Y^2)\}x\\
&\quad+\{(-2674-2631Y+109Y^2+332Y^3)-(292+82Y+4Y^2)(-7-5Y+Y^2)\}.
\end{aligned}$$

On substitution of $\frac{1}{3}(Y-x+2)$ for x^2 in this expression, powers of Y below the third necessarily disappear; the verification is superfluous except as a check, and for the calculation of the remainder we have only to examine terms involving Y^3 or Y^4. The aggregate of these is

$$(-92Y^3+8Y^4)x+(72Y^3-4Y^4),$$

and therefore

$$\frac{N}{Y^3Z^2}=\frac{3x-7}{Y^3}+\frac{x-5}{Y^2}-\frac{2x-1}{Y}+\frac{(-92+8Y)x+(72-4Y)}{16Z^2}.$$

The form of the last numerator is unnatural, and if we do not propose to break up the last fraction we shall replace Y by its value in terms of x. But if we are going to complete the resolution, we substitute for Y directly as $2Z+3x-8$, and we have

$$\begin{aligned}(-92+8Y)x+(72-4Y)&=24x^2+(-168+16Z)x+(104-8Z)\\ &=(-144+16Z)x+(32+16Z);\end{aligned}$$

whence, finally, $$\frac{N}{Y^3Z^2}=\frac{3x-7}{Y^3}+\frac{x-5}{Y^2}-\frac{2x-1}{Y}-\frac{9x-2}{Z^2}+\frac{x+1}{Z}.$$

It is perhaps unnecessary to add that I have dealt with Y before Z only to illustrate the method. If a complete resolution is known to be required, it is usually best to deal first with the factors that are raised to the smallest power.

E. H. NEVILLE.

676. [L^1. 16. a.] *Note on Angles connected with an Ellipse.*

Consider an ellipse whose eccentricity is $\cos\alpha$, then with the ordinary notation $\angle CSB = \alpha$.

At B erect a perpendicular to the plane of the ellipse and take any point Q on this perpendicular; join QC, QS. Let $\angle CQB = \lambda$, $\angle QSB = \theta$, $\angle QSC = \phi$.

Then, in the usual way, the following relations can be written down:

$$\cos\phi = \cos\alpha \,.\, \cos\theta, \quad \text{......(1)}$$

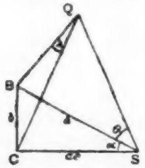
$$\sin\alpha = \tan\theta \,.\, \tan\lambda, \quad \text{......(2)}$$

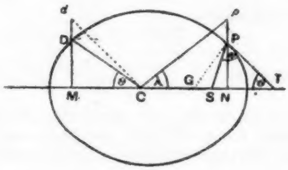
$$\tan\alpha = \tan\phi \,.\, \sin\lambda, \quad \text{......(3)}$$

$$\sin\theta = \sin\phi \,.\, \cos\lambda. \quad \text{......(4)}$$

Next choose the point P on the ellipse whose eccentric angle is λ.

Draw the tangent PT and the normal PG and let $\angle STP = \theta$, $\angle SPT = \phi$.





Let CD be the semi-diameter conjugate to CP, p and d being the points on the auxiliary circle corresponding to P and D.

Then
$$\cos\alpha = e = \frac{SG}{SP} = \frac{\sin SPG}{\sin SGP} = \frac{\cos\phi}{\cos\theta};$$

$$\therefore \cos\phi = \cos\alpha \,.\, \cos\theta. \quad \text{......(1)}$$

Again
$$\tan\theta = \frac{DM}{CM} = \frac{\frac{b}{a}dM}{pN} = \frac{b}{a}\cdot\frac{CN}{pN} = \sin\alpha \cot\lambda;$$

$$\therefore \sin\alpha = \tan\theta \,.\, \tan\lambda; \quad \text{......(2)}$$

also
$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{b}{CD\cos\phi} = \frac{b}{e\,.\,CD\cos\theta} = \frac{b}{e\,.\,CM} = \frac{b}{ae\sin\lambda}, \text{ using (1);}$$

$$\therefore \tan\alpha = \tan\phi \,.\, \sin\lambda \quad \text{......(3)}$$

and
$$\sin\theta = \frac{DM}{CD} = \frac{\frac{b}{a}dM}{CD} = \frac{b}{CD}\cdot\frac{dM}{Cd} = \sin\phi \,.\, \cos\lambda. \quad \text{......(4)}$$

A correspondence is thus established between the angles connected with the ellipse and the angles of the solid figure.

[It is well known that $\triangle$s CpN, CMd are congruent; also

$$\sin\phi = \frac{SY}{SP} = \frac{S'Y'}{S'P} = \sqrt{\frac{SY\,.\,S'Y'}{SP\,.\,S'P}} = \frac{b}{CD},$$

where SY is the perpendicular from S on the tangent PT.]

Clifton, 20th January, 1922. E. P. Lewis.

677. [K^1. 6. a.] *The Conditions for the Concurrence of three Straight Lines.*

If the lines whose equations are $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$ meet at a point, the determinant $|a_1b_2c_3|$ vanishes. This condition is generally stated or used as though it were *sufficient* as well as

necessary, but this is not the case, since the condition is satisfied by any three *parallel* lines. If the lines are to meet at a *finite* point, the additional conditions required are that the minors of c_1, c_2, c_3, *i.e.* the determinants $|a_2b_3|$, $|a_1b_3|$, $|a_1b_2|$ shall all be $\neq 0$. If the equations to the lines are written in slope form, $y = m_1x + c_1$, etc., the conditions may be conveniently stated as (i) $|1\ m_1c_2| = 0$, (ii) m_1, m_2 and m_3 all different, *i.e.* the *slopes* are all different.

The use of an incomplete criterion in a case in which it produces a correct result is very misleading to students. For an example see Benny's excellent *Plane Geometry*, § 48, where the vanishing of the determinant is said to verify the concurrence of the medians of a specified triangle; in this case the supplementary conditions though not expressed are, of course, in fact fulfilled.

Chrystal, *Algebra*, Part I. § 25, treats the matter algebraically without explicitly introducing such advanced considerations as the "rank" of the matrix of the equations. G. J. LIDSTONE.

678. [V. 1. a.] *An attempt to make Mathematics interesting.*

A common experience of every teacher of Mathematics is the problem of the student who cannot see any use for Mathematics. Teachers of History, French, English, etc., can arouse enthusiasm in every one by the intrinsic interest of their subjects: but away from the all-important personality of the teacher, I am afraid that Mathematics is often felt to be a dry as dust subject by the majority of pupils whose bent is not that way. Possibly our text-books are also a little to blame; there is little encouragement for the average student to take up a text-book for reading. And, although it may be heresy to say so, after seeing the rapt and astonished expression on the faces of our Sixth on first dipping into Thompson's *Calculus made Easy*, I for one wish there were more works written in that style.

But why should not the romance of Mathematics be felt by a class? The History teacher conjures with the names of Coeur de Lion and Bonaparte; why should not Euclid and Descartes be equally alive? What is more fascinating than the history of the Newton-Leibniz quarrel, with Gerhardt's dramatic discovery in Leibniz's own handwriting of extracts from Newton's works; or of Archimedes, with his burning glasses and triangles?

The following is an attempt of the writer to interest the average student through the medium of history.

Pressure of time prevented all but a few informal remarks during class, and absolutely forbade a series of formal lectures on the *History of Mathematics.* The plan outlined below was adopted. The form was "mixed," and the result showed that the average girl showed at least as much interest in her work as the average boy.

At the beginning of term, the following was put up on the Form notice board:

Individual Work for .... Term.

(i) Read up (make précis) in Ball's *History of Mathematics*:

(*a*) Systems of Numeration and Primitive Arithmetic, Pp. 2-9 and 121-128.
(*b*) Development of Arithmetic, „ 182-197.
(*c*) Modern Mathematics, „ 263-267.
(*d*) Non-Euclidean Geometry, „ 485-489.

And in Branford's *Study of Mathematical Education*: *

(*e*) History of Arithmetic, „ 47-74.
(*f*) Nature of Geometrical Knowledge, „ 277-294.
(*g*) Genesis of Geometry, „ 326-346.

[Branford introduces many hints on teaching. Omit these.]

* *Study of Mathematical Education* (Clarendon Press).

(ii) Consult *Encyclopaedia Britannica* under heading *Arithmetic* and *Geometry.*

(iii) From above matter, write an extended essay—not less than 1000 words —on "The Evolution of Arithmetic."

One copy of each of the above was put in the Form room and used as individuals had inclination and leisure. Another term Whitehead's *Introduction to Mathematics* was used, and selected biographies were got up from Ball's *History.*

Books alternative and perhaps preferable to the above will suggest themselves to all.

That the experiment has succeeded is beyond question. Subjects such as the Binomial Theorem, apparently leading nowhere, now suggest Newton—"the whitest soul I ever knew," to quote Bishop Burnet—little older when he invented it than the pupils themselves: and the added interest leads to added knowledge. One might quote many more examples.

Lack of space prevents more than the bare mention of the value of students reading for themselves; the primary object was to arouse more interest in the average student. D. WOOTTON.

Weymouth Secondary School.

[For some years past "Mathematics Clubs" have been a notable feature in University and College life in the United States. *The American Mathematical Monthly* has from time to time published the programmes of selected meetings, and with the kind permission of Prof. Archibald, of Brown University, we are able to extract and append to this Note a few subjects that may be suggestive to teachers who wish to make the experiment.

Flatlanders—Women in Mathematics—Mathematical Fallacies—History of Mathematics in the ——th century—The Leibniz-Newton controversy—Historical development of the number system—Trigonometrical functions and geometrical construction—Proofs of I. 47—The Circles of Apollonius—Constellations and myths about them—Mathematics, the farmer, and the weather—The monkey and coconut problem—Russian peasants and devices used in Mathematics—The Geometry of the circle or compass—Methods of counting—Isosceles triangles—Magic squares—Zeno's paradoxes of motion—Game of Kim—Simple properties of covariants and invariants—Solution of quadratics—Interpolations—The theory of complex number system—The History of the Parallel Postulate—The three famous problems of antiquity.

The above selection from one number shows the *variety* both in content and difficulty of papers and discussions that is possible in the regular meetings of a session. It should be possible by careful grading to select topics that would interest and amuse school forms much below the Sixth, and scholarship candidates who have to do "Essays" might be utilised for the purpose of reading papers or mingling in discussions. Should any desire to have a completer list, including more advanced subjects, be expressed, we shall be happy to provide it.]

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180. Wilkie, the author of the *Epigoniad*, was Professor of Natural Philosophy at St. Andrews in the days of Henry Erskine. He was a very absent-minded man. One day he met a former pupil in the streets, and addressed him thus: "I am sorry to hear, my dear boy, that you have had fever in your family. Was it you or your brother who died of it?" "It was me, sir!" "Very sorry! Dear me! I thought so; very sorry, very sorry for it!"

REVIEWS.

Fundamental Congruence Solutions. By LT.-COL. ALLAN CUNNINGHAM, R.E., and (the late) T. G. CREAK, M.A. Pp. 92. 10s. net. 1923. (Hodgson.)

From Fermat's theorem the congruence

$$x^{p-1} - 1 \equiv 0 \pmod{p},$$

p being prime, is known to be satisfied by

$$x \equiv 1, 2, 3, \ldots p-1 \pmod{p}.$$

If x denotes any one of these numbers and ξ, ν are two integers whose product is $p-1$, and if

$$y \equiv x^{\nu} \pmod{p}, \text{ then } y^{\xi} \equiv 1 \pmod{p}.$$

For each factor ξ of $p-1$ there exist integers $y_1, y_2 \ldots y_i \ldots$ such that $y_i^{\xi} \equiv 1 \pmod{p}$ and that no lower power of y_i is congruent to unity (mod p). The present volume gives *one* value of y for every possible ξ, ν so long as $p < 10000$.

ξ, ν	ξ, ν
y	y
75,76	76,75
33	40

Typical entries in the margin, under $p=5701$, mean that

$$33^{75} \equiv 1, \; 40^{76} \equiv 1 \pmod{5701}$$

and that 75, 76 respectively are the least exponents ξ, η such that $33^{\xi} \equiv 1$, $40^{\eta} \equiv 1 \pmod{5701}$. The entry under $\xi = p-1$, $\nu = 1$ is a primitive root of the congruence $x^{p-1} \equiv 1 \pmod{p}$, which could also be taken from an earlier table of Col. Cunningham's. In the present table there is nothing to show the relation of y, as given by the table, to the primitive root. Thus, under $p=5701$, $g=2$ and $33 \equiv 2^{76t}$, $40 \equiv 2^{75u} \pmod{5701}$, but the values of t, u are not stated. Though subject to this limitation the table is the result of much accurate and painstaking work: it records the results of more than 17000 numerical calculations. W. E. H. B.

Elements of Projective Geometry. By LING, WENTWORTH and SMITH. Pp. 186. 12s. 6d. 1922. (Boston, Ginn.)

There is no apparent reason why this book should either have been written or be read: it falls surprisingly below its authors' usual standard. Even the diagrams are slovenly.

Higher Geometry: an Introduction to Advanced Methods in Analytical Geometry. By F. S. WOODS. Pp. 423. 22s. 6d. 1922. (Boston: Ginn.)

A pleasant, careful and well-printed book. In 1, 2, 3, 4 and N dimensions, it gives an account, with many exercises, of both projective and metrical properties, each set of co-ordinates being introduced as belonging to objects other than points in the next lower space. Thus the Part on four dimensions starts with chapters on the co-ordinates of straight lines and spheres in ordinary space, led up to by some pretty work on cyclides and cylindroids in the earlier Parts.

The choice of topics is restrained, and what is admitted is treated thoroughly enough to be interesting: except possibly the short section on contact transformations might with advantage have been either expanded or omitted. The definitions and explanations of imaginary and infinite elements are exceptionally clear, bringing out their dependence on the co-ordinate system. H. P. H.

Practical Mathematics. By V. T. SAUNDERS, M.A. Pp. 46. 1s. 6d. 1923. (Bell.)

Introduction to Practical Mathematics. By V. S. BRYANT. Pp. 77, with 12 pp. of Revision papers, and Answers. 2s. 6d. (Clarendon Press.)

The first book should prove useful; with the professed purpose of teaching no more than accurate measurement and calculation, the pupil is incidentally

introduced to most of the ideas and instruments used in the elementary Physics laboratory. Indeed, the title is rather an unfortunate description of the contents; for we are accustomed to attach another meaning to the term *Practical Mathematics*. This work is really concerned with what is usually termed the Mathematical Laboratory; and a better description would have been Laboratory Mathematics.

The second book is more of the class we should usually include under the term *Practical Mathematics*. Each experiment is made the groundwork of exercises on the manipulation of formulae. The general scheme is good, and the Revision Papers at the end leave nothing to be desired except that there might be more of them. J. M. C.

A History of Greek Mathematics. By SIR THOMAS HEATH, K.C.B., K.C.V.O., F.R.S. Vol. I. *From Thales to Euclid*. Pp. xv + 446. Vol. II. *From Aristarchus to Diophantus*. Pp. xi + 586. 50s. net. 1921. (Clarendon Press.)

If Sir Thomas Heath were to write no more, and as long as he is in our midst that is unthinkable, he has already deserved more than well of his generation. He has never faltered in his allegiance to the great pioneers. To him it has been a labour of love to assist in making broader and deeper the foundations of their age-long fame. He has given us the classical edition of Euclid's *Elements*. His *Diophantus* and *Apollonius* both filled gaps the very existence of which had been unsuspected, forgotten, or ignored by the majority of mathematicians in the Britain of fifty to sixty years ago. The first English *Diophantus*, which appeared in 1885, opened to the English reader a field almost untouched in these islands since the days of John Pell. The edition of 1910 revealed for the first time to many the work of Euler and Fermat. In the case of the Apollonius we had less reason to blush for our indifference or indolence. The dedication of our author's *Apollonius of Perga*—Manibus Edmundi Halley—reminded us that in the first decade of the eighteenth century the successor of Wallis and Flamsteed had given to the world the first four books in Greek with a Latin translation, and had not only translated the next three into Latin from the Arabic, but had added a conjectural restoration of the missing eighth book. The *Apollonius* of 1896 brought home to us again the importance in the history of the science of a work then two thousand years old, and dealt fully and faithfully with the researches of continental investigators such as Heiberg, Cantor and Zeuthen. Next came the *Archimedes* of 1897, and *Aristarchus* followed five years after the *Euclid*. To write these volumes and minor works was, in the old sense of the words, the avocation, the amusement of one immersed in the work of a Civil servant, and steadily rising to the important office he now holds at the Treasury. The two volumes before us are a fitting crown to the labours of so many years. Not that we may not have more from the same busy pen—we have heard whispers of further researches in connection with the debt of science to Aristotle, ὁ δαιμόνιος, "the daemon of nature," and of a sifting of the gossipy records of Jamblichus. But the *Euclid* and the *History* it will not be easy even for Sir Thomas Heath to surpass. In the *History* he has kept strictly to the lines laid down in the preface. Questions of cost and space had no doubt to be considered. The most tempting of discursions have been mercilessly omitted. The Johnsonian adage has been borne in mind:—the great excellence of a writer is to put into his book as much as it will hold. As old Fuller has it: he who expects what in reason he cannot expect—may expect. So those who lament the absence of the discursive matter must take what comfort they can from the reflection that the man who can write the best small book on a subject is he who has first written a big one on it.

As a preliminary word of advice to the student we may quote the caution given by the old translator of the fables of Bidpai: "He that beginneth not to read this book from the beginning to the ende, and that advisedly followyth not the order he findeth written, shall never profit anything thereby. But reading through and oft, advising what he readeth, hee shall find a merveylous benefit thereof."

The story here unfolded is as fascinating as it is marvellous. "If anything

could enhance the marvel of it," says the author elsewhere, "it would be the consideration of the shortness of the time (about 350 years) within which the Greeks, starting from the very beginning, brought geometry to the point of performing operations equivalent to the integral calculus and, in the realm of astronomy, actually anticipated Copernicus." The introductory chapter gives a bird's eye view of the conditions amid which the Greek genius was fostered and became capable of creating the sciences as they did create them, a summary of the mathematics they possessed and the part that it played in the education of the young. The chapter on Greek numerical notation and arithmetical operations is a natural preface to that on the Pythagorean Arithmetic, just as that on the origins of Greek geometry from Thales onwards is a necessary preliminary to the chapter on Pythagorean geometry. We note *en passant* that in the section on the tradition connecting Euc. I. 47 with the name of Pythagoras, the conclusion arrived at by the author is that he cannot "go so far as to deny to Pythagoras the credit of the discovery." and he continues, "nay, I like to believe that the tradition is right, and that it was really his." "If the Indians really achieved a scientific proof, there is neither evidence nor probability that the Greeks are indebted to India for theirs." Chapter VI. deals with progress in the elements down to the time of Plato, and is followed by some sixty pages discussing the history of the three famous problems of antiquity. A discussion of Zeno's arguments about motion precedes a great chapter on Plato. Chapter X., "From Plato to Euclid," brings in Eudoxus and Aristotle. The last ninety pages of the volume are devoted to Euclid.

Vol. II. opens with a short chapter on Aristarchus "the mathematician," "the ancient Copernicus." Archimedes, "who shall be believed in whatever he shall affirm," and Apollonius have each about 100 pp. The next chapters deal with the successors of the great geometers, with "some handbooks," and with the development of Trigonometry, plane and spherical. Sixty pages are given to Mensuration and Heron of Alexandria. To Pappus are allotted 80 pages, to Diophantus 70. The text is brought to a close with some forty pages on "Commentators and Byzantines," and last, but not least, are two excellent indexes, one English and the other of Greek words.

The whole is arranged according to subjects. Even from this inadequate epitome it will be gathered that although the treatment is by subjects, the author has avoided the danger of dwarfing the great outstanding figures, of obscuring their amazing versatility, of losing sight amid a mass of detail of the genesis of form and style with their development into perfection. The pages of these volumes are further enriched by the effort made by Sir Thomas Heath to enable his readers to see for themselves the procedure by which the results were obtained. "I have taken great pains, in the most significant cases, to show the course of the argument in sufficient detail to enable a competent mathematician to grasp the method used and to apply it, if he will, to other similar investigations. "And thus we behold not only the thinker *serit arbores quae alteri saeculo prosint* but generation after generation, each performing its function of receiving the principles of knowledge, widening their application, enlarging their number, *et quasi cursores vitai lampada tradunt.* Nor do we fail to recognise that these acute and original minds had serious limitations. The discovery of incommensurable magnitudes was eventually to raise a barrier to progress—a barrier not to be removed until the coming of Descartes. Their lack of generalising power led to great waste of energy. It was seldom that they attained general points of view, or were able to comprehend a number of previously disconnected results in the grasp of a general method. "For instance," says Prof. Lamb, "it was long before the Greek mathematicians rose to the general conception of magnitude as distinguished from magnitudes of particular kinds; and it is recorded that a certain theorem in proportion was wont to be proved separately for numbers, lines, solids, and times, because no term had been invented which would comprehend these things. . . ." So again although Diophantus used symbols, he does not seem to have grasped the power that symbolism gives of carrying out mechanically a chain of reasoning. On the other hand, the distinction drawn by the Greeks between "numbers" and "magnitudes" or "quantities"

was a distinction which became gradually obliterated as people found the convenience of identifying numbers with measures of lengths, and the process was completed with the invention of analytical geometry. "It is in comparatively modern times that mathematicians have adequately realised the importance of this logically valid distinction made by the Greeks. It is a curious fact that the abandonment of strictly logical thinking should have led to results which transgressed what was then known of logic, but which are now known to be readily interpretable in the terms of what we now know of logic."*

The science of the ancients is full of enigmas, and for the answers we may have to depend on mere fragments of documents. Nor is it easy to decide if what they contain is mere tradition, and even if genuine copies how far they are corrupt. Hence the qualifications demanded of the historian of the subject are of an exceptional order. He must have the subject at his fingers' ends, and from the nature of the sources from which our knowledge of its development is derived he must have among his many gifts the gift of intuition. He may have to reconstruct the solution of a problem to which we have a mere allusion: to fix the date of a work mentioned by but one or two writers long after it was written: to discover the real author, known or unknown, of the few sentences that are all that is left to us of some treatise: to decide upon the meaning of a text which the copyists have left in an unintelligible form. Even with the divining power at its highest there is no royal road to success. Pierre Duhem has suggested that what the historian requires most of all is the possession in a very eminent degree of what he calls the faculty of divination. Texts must be subjected to the minutest examination: every atom of relevant evidence must be weighed in the scales: comparison must be made with the state of contemporary knowledge, and this calls for the widest reading and the keenest insight. He considers that Paul Tannery often exercised this faculty with conspicuous success, but that, emboldened with impunity, he more than once took a leap in the dark—and fell. He had the defects of his qualities. Fortunately there is nothing more stimulating than contact with original sources: works of genius have in themselves an interest that never fails. In Sir Thomas Heath we have, as Erasmus said of Tunstall, a scholar who is *dictus ad unguem*. In these volumes we see as occasion arises the evidence displayed, relative values discussed, and probabilities cautiously weighed. Then comes the impartial summing up with the unimpeachable verdict. Sound judgment is a priceless possession, and more than once in studying these pages we have been reminded of one of Newton's letters:—"And now I have told you my opinion in these things I will give you Mr Oughtred's, a Man whose judgment (if any man's) may be safely relied upon."

The practised skill of the veteran craftsman has in these thousand pages interwoven such biographies as were possible and the analyses that tell us what we owe to the Greek pioneers. His narrative of their magnificent conquests is at once accurate and succinct. He confines himself strictly to the business in hand. To some of us, no doubt, gossipy digressions are the honey dew and milk of paradise, but they have no place in what is a miracle of extended exposition coupled with wise compression. Clear and massive stand out the great figures: we cannot fail to join with him in homage to those who led men forth from the Cimmerian darkness that lay beneath "the clouds of unknowynge." We see how from exact thinking was laid the foundation upon which arose the superstructure of so much of later science. We see the inspiration of genius at its work of welding and vitalising. That is the main reason why such works as these should be within the reach of every serious student, and of all who are interested in the development of man's intellectual faculties. The student is laying the foundations of a critical sense as he learns the manner in which the difficulties of a discoverer have been evaded or surmounted: his own imagination is quickened as he feels with another the exaltation of successful achievement: his horizon is widened as the futility of dogmatism dawns upon him, as he grasps the relative nature of knowledge and the insignificant nature of the part played by the average individual.

* *The Nature of Mathematics*, p. 31, P. E. B. Jourdain.

And beneath all the chances and changes he discerns the patient, slow work of continuous selection. Many classic instances might be given of the sluggish rate at which new doctrines are accepted. It is startling to think that Milton, who, in the land where "flattery and fustian" reigned supreme, had stood face to face with Galileo and had heard his revolutionary doctrines from the lips of the prisoner of the Inquisition, is found years after teaching his pupils the Ptolemaic theory and instructing them in the Manilian astrology. Even more extraordinary it is to find that 200 years after the death of Copernicus an edition of the *Orbis Pictus* of Comenius taught that the Earth is the centre of the solar system.

A handful of men in some three centuries made *ab initio* such discoveries and such developments as remained unparalleled for ages to come. Then a twilight descended upon the scene. The work of the Greek mathematicians of the next thousand years can be told in thirty pages. The dawn was not to break until an Englishman at Oxford wrote in 1621 :—" In the most fair frame of geometry there are two defects, two blots." But the wonder remains. As our author elsewhere reminds us, the Greeks had formulated the principles, settled the terminology, and invented the methods "with such certainty that in the centuries that have since elapsed there has been no need to reconstruct, still less to reject as unsound, any essential part of their doctrine." To Greece we look for the most signal example of

> Man, the marvellous thing, that in the dark
> Works with his little strength to make a light.

That little strength derived its astounding energy from the innate craving of the Greek intellect for regularity, simplicity, and symmetry—for beauty.

We cannot close this notice without quoting the last words of the Preface :—

"The work was begun in 1913, but the bulk of it was written, as a distraction, during the first three years of the war, the hideous course of which seemed day by day to enforce the profound truth conveyed in the answer of Plato to the Delians. When they consulted him on the problem set them by the Oracle, namely, that of duplicating the cube, he replied, 'It must be supposed, not that the god specially wished this problem solved, but that he would have the Greeks desist from war and wickedness and cultivate the Muses, so that, their passion being assuaged by philosophy and mathematics, they might live in innocent and mutually helpful intercourse with one another.' Truly

> Greece and her foundations are
> Built below the tide of war,
> Based on the crystálline sea
> Of thought and its eternity."

New Mathematical Pastimes. By Major P. A. MacMahon. Pp. vii + 116. 12s. net. 1921. (Cam. Univ. Press.)

"Problèmes Plaisants et Délectables" more than three centuries ago moved a friend to address their compiler, Claude-Gasper Bachet, Sieur de Méziriac, in rapturous terms :—

> "Tes jeux, mon cher Bachet, doctement inventés
> Savent bien accoupler d'un art inimitable
> Le plaisir au profit. . . ."

and after exclaiming, "O que de beaux secrets," he proceeded :

> "Qui se paîtront souvent de tes fameux écrits
> Consacreront ton nom au temple de mémoire."

Had a Sylvester been permitted to read the proof-sheets of Major MacMahon's volume, he might have been inspired to a similar flight, couched in terms more consonant with the facts and with modern ideas of the appropriate. The author himself has enriched his pages with a number of poetical quotations, gathered from a wide field, often ingeniously apt, and descriptive of the nature or the treatment of the subjects in the sections they head. It is clear that "cheerfulness was always breaking in" as Major MacMahon strolled along this "pleasant bye-path of mathematics which has almost entirely escaped

the attention of the well-known writers upon mathematical recreations and amusements." And so it will probably be with the reader as he follows his guide.

The first part, of 50 pp., is succinctly described as "generalised dominoes." The reader is presented with "sets of triangles, squares, etc., which, as regards a particular set, are all the same size and shape, but are differently coloured or numbered : . . . . a particular set of pieces may be set up into a square, rectangular, hexagonal or other shape so that certain contact laws inside the boundary of the figure are satisfied." Thus we have in turn Equilateral Triangle, Square, Right-angled Triangle, Cube, and Regular Hexagon Pastimes showing vast possibilities and of types numerous enough to suit the most exacting; some are easy, some difficult, of some we are told, "it is not known how many of these exist," and of others that they "are by no means easy, but all are believed to be possible." It is quite clear that there is plenty of room here for the exercise of untold patience and of considerable ingenuity.

Part II. (30 pp.) deals with the transformation of sets of pieces of the same shape and different colours into sets of pieces of different shapes but of the same colour. Here be varieties to suit every taste—some "very interesting, but the pieces difficult to assemble," while with others "the reader will probably have no difficulty," though the author admits of one that "he has not particularly examined it, but he recommends it with confidence."

Part III. is heralded by the Tupperian line :—"The story without an end that angels throng to hear," the story being that of repeating designs for decorative work, proceeding from the simple shapes to the quest for any shape that will cover any flat surface completely by simple repetition. The author proposes in the near future to publish a more general treatment of the subject matter of Part III. Meanwhile those who have amused themselves by building up some of the more complicated of the repeating patterns indicated in this delightful volume may find it interesting to discover if they have been anticipated by the artless compilers of the Grammars of Ornament and Anatomies of Pattern of days gone by. To such volumes there is no reference in the Bibliography supplied by the author, which, however, is a most useful chronological catalogue of works on Mathematical Recreations. W. J. G.

Researches in Geometry. By DR. P. S. G. DUBASH, D.Sc. 8 annas. (Taraporewalla & Sons, Bombay.)

This brochure embodies the results of the mathematical diversions of an author whose chief activities seem to be exercised on other subjects. They concern circle-squaring, angle-trisecting, and the inscription in a circle of various refractory regular polygons. Criticism, in spite of the ambitious title, is to some extent disarmed by the very modest preface, by certain naïve remarks in the text, and by occasional admissions as to the merely approximate correctness of the constructions. We regret to find that these admissions are not made throughout. We have two rectifications for a semicircular arc, in one which π is taken as 3·14 and in the other as $1+\sec 62^\circ\ (=3\cdot130$ to 3 decl. places). For the trisections we have :

(i) A special method for 60° which gives $\frac{1}{5}\sqrt{3}$ for the tangent of 20°, that is 19° 6′ to the nearest min. for each of the extreme third parts, if carried out correctly; there is no warning except the discrepancy in the diagram between the middle part and each of the extremes—it amounts to about 3°—that the method is only approximate. The construction appears to be based on the fallacy that if, in a triangle ABC, $BC=2CA$, then $A=2B$.

(ii) a general method for 'trisecting' (or n-secting) any angle up to 54°, involving the fallacy $\sin\theta/3=\frac{1}{3}\sin\theta$ (and generally $\sin\theta/n=(1/n)\sin\theta$). In both cases a little work on the author's part and the use of Trigonometrical Tables, with which he appears to have some acquaintance, would have enabled him to state approximately the amount of error involved in the use of these constructions by any one who wanted them for a practical purpose.

Coming to the regular pentagon he gives two constructions for 72°, viz. $\tan^{-1}3$ and $45^\circ+\tan^{-1}.5$. Each of these equivalent constructions gives 71° 34′ to the nearest minute. Hence they are less accurate than two fairly well known methods of the text-books on Practical Geometry—that of De Ville

(1628), in which the diameter AB of a circle having been divided at Q so that $AQ = \frac{2}{5} AB$ and having had an equilateral triangle AKB described upon it, KQ is produced to cut the circumference in F, and that of A. Dürer (1525), where 3 equal circles are described each with its centre at the intersection of the other two. De Ville's is very accurate, giving the angle subtended at the centre by AQ as 71° 57′. Dürer's value is 71° 38′, so that Dr. Dubash is in good company. His method has one advantage, simplicity, which might cause it to be preferred even to De Ville's if the degree of accuracy was considered sufficient. But what are we to think of the following statement: "*In* 90 *per cent. of cases the angle comes out exactly* 72°, sometimes it is less. So for practical purposes it is quite good, but *the theoretical mathematicians will have difficulty that it should ever be* 72°." As a matter of fact the figure in the text is not one of the 90 per cent. but gives about 71° 30′.

We have often had to deal, when constructing models of star-polyhedra, with the famous triangle of Euclid IV. 10, and have generally preferred to use simple approximate methods to the correct but more complicated one based on Euclid II. 11. We have found very helpful successive 'numbers of Metius' (1571-1635) in the series 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, etc., in which after the first two, each term is found by adding together the two immediately preceding it. Thus if we construct isosceles triangles with sides (34, 34, 21), (55, 55, 34) (89, 89, 55)—the base angles will be respectively 72° 0′ 42″, 71° 59′ 44″, 72° 0′ 6″. In the second and third the degree of accuracy is beyond that required in practice, instrumental and other errors being probably greater than those due to the method used.

A construction for a regular heptagon, which the author asserts to be correct, is easily seen to give $\tan^{-1}\frac{1}{3}(2+\sqrt{3})$ for $\frac{1}{7}$ of 360°, which to the nearest minute is 51° 12′, and is therefore about 14′ in defect of the true value. The method is, however, simple and may be practically useful in design. It does not yield quite such a close approximation as one given by Leonardo da Vinci (1451-1519) and once known as the *Indian rule.* The latter takes for the side of the inscribed heptagon the perpendicular from the centre to that of the hexagon, thus giving for the angle at the centre $2\sin^{-1}\frac{1}{4}\sqrt{3}$, *i.e.* 51° 19′. The construction for the regular nonagon is founded on the faulty trisection of 60° already noticed. For 40° we get about 40° 54′. De Ville's method, in which $AQ = \frac{2}{9} AB$, gives 40° 17′. We have checked our calculations for the angles subtended at the centre by the chord taken as a side in the case of De Ville and Dürer by the results given to nearest second in an article by Mr. A. J. Pressland in *Proc. Edin. Math. Soc.*, pp. 23-34, Vol. X., "On the History and Degree of certain Geometrical Approximations," which contains much interesting historical matter.

Linear Geometry. A Difficult Problem. By EPHPHETA. (Typed copy.)

"During the study of non-Euclidian Geometry I happened to find some facts connected with elementary Geometry which I believe hitherto unknown and it is exactly to make sure about this that I trouble you as a means of enquiring about it, trusting some of your readers will be so kind as to enlighten me on the matter."

To construct, inscribed in a circle, an isosceles triangle which has been previously drawn with its sides in the ratio $\sqrt{7} : \sqrt{7} : \sqrt{3}$ under the conditions:

(i) the solution must not be the result of any calculus—algebraical or graphical;

(ii) the centre must not be found by drawing perpendiculars to the sides, nor by arcs with centres at their end points;

(iii) the sides must be the last lines drawn.

As to (i) we confess our inability to understand it and the consequent necessity of gathering from the author's own solution what theorems or what constructions we may use.

He states that the solution requires the trisection of an angle, which he proceeds to effect as follows:

"Let AOB be an angle subtended at the centre O of a circle ABC by the arc AQB and the chord APB. By means of a *moveable ruler* always passing

through O and a pair of compasses with one point at A, find *by trial* a position of the ruler in which the points P and Q in which it crosses the chord and the arc respectively are equidistant from A: then AOQ is one-third of AOB."

We agree, but we need scarcely point out that he is not keeping within the bounds laid down in the postulates of Elementary Geometry for a solution by "ruler and compasses."

He now proceeds to show how, granted his method of trisection, he can when the three vertices of the required triangle have been determined, find a fourth point which lies on its circumcircle, incidentally demonstrating a theorem which he supposes new, although only a special case of a known general theorem. But any number of such points may be found without any trisection, as shown below:

With O, A, B, C as above take *any* point Q on the circumference of the circle ABC and an arc $QR = QB$. Join OQ, AR, and let them meet, produced if necessary in J. Then J lies on the circumcircle of AOB. For $J\hat{A}B = \frac{1}{2} R\hat{O}B$ on the same arc BR.

$\therefore$ either $J\hat{A}B = J\hat{O}B$ or $J\hat{A}B + J\hat{O}B = 2$ right angles: thus J is obtained without a trisection or even a bisection, and requires no more calculus of any sort than *Ephpheta's* construction.

If it happened in addition, as *Ephpheta* supposes it to have done, that arc RQ = arc RA, J would be a point of trisection of the arc AJB of the circumcircle of AOB, as also would I be found by drawing OR, BQ; since then $B\hat{O}J = J\hat{O}I = I\hat{O}A$. It is this result which he has enunciated thus:—

"Etant données deux lignes polygonales regulières à trois cotés ($ARQB$, $AIJB$) inscrites dans deux segments de circonférences à base commune, le centre d'un des cercles étant placé sur l'autre cercle, les prolongements des cotés extrèmes de la ligne polygonale enveloppée passent par les sommets moyens de la ligne enveloppante." This is, however, merely a special case of a known and fairly obvious more general theorem on *any* two intersecting circles with a common chord AB. If through A any straight line AQJ be drawn cutting the circles in Q, J, then the arcs BQ, BJ are corresponding parts of the two circles considered as similar figures.

Returning to the problem, a fourth point J having found on the circumcircle of the required triangle its centre may be found by drawing the perpendicular bisectors of two of the lines joining it with the angular points already found.

The problem seems of no great difficulty, and we have found several solutions which seem simpler than the above and to observe the conditions laid down more rigidly.

E. M. LANGLEY.

THE YORKSHIRE BRANCH OF THE MATHEMATICAL ASSOCIATION.

A JOINT meeting of the Yorkshire and Manchester branches of the Mathematical Association was held on Saturday, June 16th, at the Hebdon Bridge Secondary School. The meeting was the first of its kind attempted, and proved a complete success, largely owing to the excellent arrangements made for the entertainment of the two branches by the Head Master, Mr. M. Wager.

Dr. W. P. Milne, Professor at Leeds University, read a paper on the teaching of Infinite Series for the Higher School Certificate Examination, and Mr. G. St. L. Carson, Inspector of Secondary Schools in the Manchester district, one on the Teaching of Geometry. After the papers, Yorkshire and Manchester Mathematicians made new friendships and compared experiences over tea, and expressed a resolve to meet again in a similar way next year.

THE LIBRARY: CHANGE OF ADDRESS.

For some years past the cost of accommodating the Library on premises situated centrally has interfered with maintenance and has prohibited outlay on extension. The records for the same period show that easy access to the shelves has been of very little value to members of the Association. The books have therefore been removed to

160 Castle Hill, Reading,

where they are in charge of Prof. E. H. Neville, who has undertaken to serve as Librarian.

Members who wish to consult the books there will be very welcome, but the hope which it is most reasonable to entertain is of developing a postal lending library of mathematical works on the lines of the London Library. Members are invited to express their needs of books to the Librarian *without reference to the Catalogues*:

(i) In order that the Council may gauge the strength of the demand for a library of this kind and discover in what directions expansion would be most useful;

(ii) Because the Catalogues are out of date;

(iii) On the chance that the Librarian may be able to put a member wishing to borrow a work not in the Library into touch with a member able to lend it.

A book that is available will be sent on the conditions:

(i) That the borrower pays expenses, including for a book that is rare or old such registration or insurance as the Librarian thinks fit;

(ii) That the book shall be returned (*a*) if the property of the Association, at any time after one month if required by another reader, and in any case before the expiration of three months, (*b*) if not the property of the Association, within whatever interval the owner or the Librarian may stipulate;

(iii) That the borrower accepts liability for any damage which a book may sustain before its receipt by the lender;

(iv) That no borrower may hold more than three books at one time, a work issued in a number of volumes being counted as one book for the purposes of this rule, but individual volumes of a periodical or of collected works being reckoned separately.

Application for a book is understood to imply agreement with these terms.

ADDITIONS.

The Librarian reports the following gifts from Mr. Greenstreet:

(E. A. Abbott)	Flatland	1884
A. M. b. O. Abenbeder	Compendio d'Algebra, Trans. (from Arabic into Spanish) and ed. by J. A. S. Pérez	1916
A. L. Baker	Elliptic Functions	1890
P. Barlow	New Mathematical and Philosophical Dictionary	1814
Jean Bernoulli	Opera Omnia (4 vols.)	1742
M. Bocher	Einführung in die Höhere Algebra, Trans. (from English into German) by H. Beck	1910
O. Bolza	Variationsrechnung. Parts 2 and 3	1909
	(Has any member the missing Part 1 to give?)	
E. Borel	Algèbre (2 vols.)	1903
	Trigonométrie	1904
	(These are elementary school-books.)	

R. J. Boscovich	Natural Philosophy. Latin text with English translation and a memoir by J. M. Child	1922
C. Bourlet	Leçons d'Algèbre élémentaire	1903
K. C. Bruhns	Manual of Logarithms	1922
Buonaventura Cavalieri	Trattato della Sfera	1682
E. Cesaro	Lehrbuch d. Algebraischen Analysis v. d. Infinitesimalrechnung	1904
A. Clebsch	Leçons sur la Géométrie, trans. (from German into French) by A. Benoîst (2 vols.)	1879, 1880
J. L. Cowley	Appendix to Euclid. 2nd ed.	
G. Cramer	Analyse des Lignes Courbes Algébriques	1750
L. Cremona	Opere Matematiche (3 vols.)	1914, 1915, 1917
S. Cunnington	Story of Arithmetic	1904
M. Dauzat	Eléments de Méthodologie Mathématique	1901
B. Donn	New Introduction to the Mathematicks	1758
J. M. C. Duhamel	Cours d'Analyse (2 vols. in one)	1841, 1840
H. Durege	Elemente der Theorie der Funktionen	1906
	Theorie der Elliptischen Funktionen	1908
W. Emerson	Elements of Geometry	1763

Mathematical Gazette, No. 140.

This was the number (May, 1919) which contained Prof. Bryan's Tables. The stock is exhausted, and the Librarian will be glad to buy back copies at their face value, or to send in exchange the Tables in the independent form in which they have since been issued by Macmillan.

Donations of back numbers of the *Gazette* are always welcome.

E. H. Neville.

NOTICE.

The Editor is unable to identify the author of an unsigned article entitled *The Epipedon*, and will be glad to have his name and address.

He will also be glad to receive his bound copy of Fiedler's Edition of Salmon's *Conics* (lent to a member some years ago), as he has been asked to lend it to some one else.

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THE MATHEMATICAL ASSOCIATION.

(An Association of Teachers and Students of Elementary Mathematics.)

"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament thereunto."—BACON.

THE MATHEMATICAL ASSOCIATION, which was founded in 1871, as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

Its purpose is to form a strong combination of all persons who are interested in promoting good methods of teaching mathematics. The Association has already been largely successful in this direction. It has become a recognised authority in its own department, and has exerted an important influence on methods of examination.

The Annual Meeting of the Association is held in January. Other Meetings are held when desired. At these Meetings papers on elementary mathematics are read and discussed.

Branches of the Association have been formed in London, Southampton, Bangor, and Sydney (New South Wales). Further information concerning these branches can be obtained from the Honorary Secretaries of the Association.

"The Mathematical Gazette" (published by Messrs. G. BELL & SONS, LTD.) is the organ of the Association. It is issued at least six times a year. The price per copy (to non-members) is usually 2s. 6d. each. The *Gazette* contains—

(1) ARTICLES, mainly on subjects within the scope of elementary mathematics;

(2) NOTES, generally with reference to shorter and more elegant methods than those in current text-books;

(3) REVIEWS, written by men of eminence in the subject of which they treat. They deal with the more important English and Foreign publications, and their aim, where possible, is to dwell on the general development of the subject, as well as upon the part played therein by the book under notice;

(4) SHORT NOTICES of books not specially dealt with in the REVIEWS;

(5) QUERIES AND ANSWERS, on mathematical topics of a general character.